The Laws of Flow

1. Introduction

Blood flow in highly branched circulatory system of elastic vessels with pulsating flow driven by the rhythmic action of the heart is very complex. However, analysis of simple models of a system made of pipes (vessels) with stiff walls and circular cross-section in the conditions of continuous flow, permits understanding of fundamental relationships between parameters used for the description of flow in real systems. From the laws describing the fluid flow, the following ones will be discussed:

- the equation of continuity,
- the Hagen-Poiseuille law,
- the Bernoulli equation.

Also the phenomena of the turbulent flow and the idea of the vascular resistance will be explained.

The goal of this laboratory exercise is to examine, using the computer simulation, the qualitative and quantitative relationships between the following quantities: the rate of flow, viscosity of fluid, pressure and vascular resistance.

2. The Equation of Continuity

If the following conditions:

- walls of vessels are stiff
- flowing fluid is incompressible
- flow is laminar and continuous

are fulfilled, then:

the rate of fluid flow Q (i.e. the volume ΔV of fluid flowing in unit time Δt, $Q = \frac{\Delta V}{\Delta t}$, via the cross section area of the vessel) in every part of the vessel is the same and constant:

$$Q_1 = Q_2 = \text{const.} \quad (1)$$

The above statement is called the law of continuity.

The equation of continuity implies a certain relationship between the average peed v of the fluid flow in a given part of the vessel and the vessel's cross-section area A in that part. The volume element ΔV (Fig. 1) can be expressed as:

$$\Delta V = A \times \Delta l \quad (2)$$

where: Δl *stands for the „length" of the volume element* ΔV.

Thus equation (1) can be rewritten in the following form:

$$\frac{A_1 \cdot \Delta l_1}{\Delta t} = \frac{A_2 \cdot \Delta l_1}{\Delta t} = \text{const.}$$

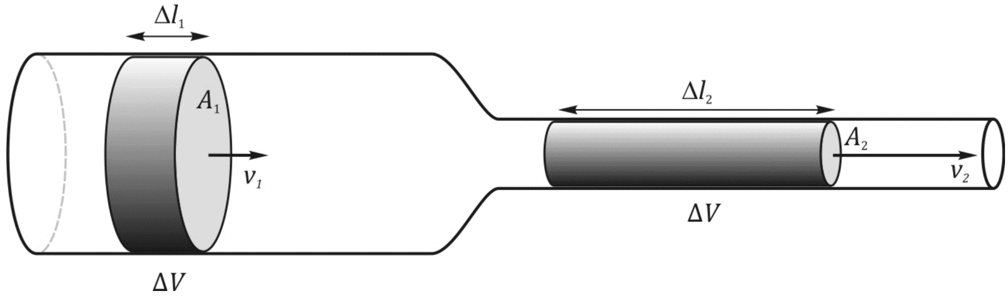

Fig. 1. The average speed of flow increases ($v_2 > v_2$) as the cross-section area of the vessel decreases ($A_2 < A_1$).

Taking into regard that $\frac{\Delta l}{\Delta t} = v$, we get:

$$A_1 \cdot v_1 = A_2 \cdot v_2 = \text{const.} \tag{3}$$

where v_1 and v_2 stand for the average speed of flowing fluid in parts of A_1 and A_2 cross-section area respectively. The above expression is another form of the equation of continuity and it can be described as follows:

For any fluid flowing through a vessel, the product of the average speed of flow v and the vessel cross-sectional area A, for a given flow, remains constant.

The main conclusion following from the above statement is that for any given flow of the incompressible fluid, the average speed of flow is inversely proportional to the cross-section area, thus it is greater in narrow parts and lesser in wider parts of a vessel. This conclusion is valid both for a single pipe and for complex, branched systems. Measurements of the speed of blood flow in different parts of the human circulatory system are consistent with the law of continuity. In the ascending aorta of 4 cm^2 cross-section area the average speed of blood flow takes the highest value of about 20 cm/s. The largest total cross-section area of the system of capillary vessels is about 2700 cm^2 and, as a result, the average speed of flowing blood is there the smallest and takes about 0.03 cm/s in a single capillary.

3. Bernoulli's Equation

Blood pressure is one of the major quantities used in diagnostics of the circulatory system. To sustain proper pressure of blood, the heart must continuously perform its task of a force pump. In the case of vessels of stiff walls and in the absence of flow resistance (no viscous friction), the work done by the pump causes an increase in the potential and kinetic energy of a volume element ΔV of the flowing fluid (Fig. 2).

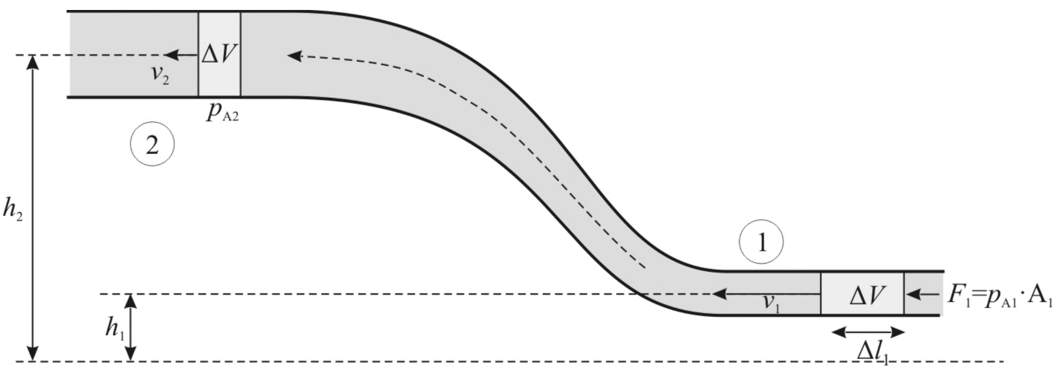

Fig. 2. To increase the mechanical energy of the fluid (the kinetic and potential energy) a force F_1 in section 1 (on the vessel's right side) has to be applied along the distance Δl_1. The amount of work W done by force F_1 equals: $W = F_1 \cdot \Delta l_1 = p_{A1} \cdot A_1 \, \Delta l_1$, where p_{A1} is the pressure in section 1 of the vessel of a cross-section area A_1.

Bernoulli has found that the total pressure in a stream of flowing fluid is a sum of three factors:

- pressure p_A resulting from the force F applied over the cross-section area A of the stream,
- pressure $p_h = \rho \cdot g \cdot h$ – resulting from increased potential energy of flowing fluid,
- pressure p_d resulting from the motion of the fluid, called the dynamic pressure given by the formula:

$$p_d = \frac{1}{2} \cdot \rho \cdot v^2 \tag{4}$$

where ρ – the density of fluid,
v – the average speed of fluid flow.

(the dynamic pressure can be for instance felt as a push exerted on a hand placed in the stream).

Thus, consequently, one may express the total pressure p_T as follows:

$$p_T = p_A + p_h + p_d \tag{5}$$

or:

$$p_T = p_A + \rho \cdot g \cdot h + \frac{1}{2} \cdot \rho \cdot v^2$$

Moreover, if:

- there is no viscous friction (viscosity of the fluid is negligible)
- the vessels' walls are stiff
- the fluid is incompressible

the total pressure has the same value in every place of the vessel :

$$p_{T1} = p_{T2} = \text{const.} \tag{6}$$

or:

$$p_{A1} + \rho \cdot g \cdot h_1 + \frac{1}{2} \cdot \rho \cdot v_1^2 = p_{A2} + \rho \cdot g \cdot h_2 + \frac{1}{2} \cdot \rho \cdot v_2^2 = \text{const.} \tag{7}$$

The above formula is known as the Bernoulli equation. If the fluid does not flow ($v = 0$, thus $p_d = 0$) the existing pressure is the sum of p_A and p_h. This sum is called the *lateral* or static pressure p_L:

$$p_L = p_A + p_h \tag{8}$$

The lateral pressure is exerted by the fluid on the vessel's walls and on surfaces of objects immersed in the fluid. Bernoulli's equation can be rewritten in a more confined form:

$$p_T = p_L + p_d \tag{9}$$

which states that the total pressure p_T existing in the stream of flowing fluid is the sum of the lateral and dynamic pressures. For horizontal tubes, where $h_1 = h_2$, equation (7) simplifies to:

$$p_{A1} + \frac{1}{2} \rho \cdot v_1^2 = p_{A2} + \frac{1}{2} \rho \cdot v_2^2 = \text{const.} \tag{10}$$

Summing up, one may say that the pressure exerted on the walls of a tube or the surface of any object that the fluid passes over is low when the speed of the fluid is high, and high when the speed is low.

Value of the total and lateral pressure can be measured directly by placing manometric tubes of the appropriate shape (Fig.3) in a stream of flowing fluid. Tubes A and C with their openings facing the stream measure the total pressure p_T whereas tubes B and D with their openings tangential to the lines of flow measure the lateral pressure p_L. The height of the fluid columns in tubes is a measure of particular pressures in the stream. The dynamic pressure can be determined indirectly by direct measurement of the total and lateral pressures as follows from equation (9):

$$p_d = p_T - p_L \tag{11}$$

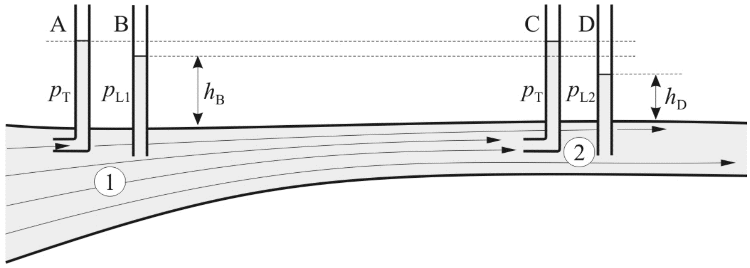

Fig. 3. The height h of fluid in the manometric tubes is proportional to the current pressure.

Bernoulli's equation permits prediction of changes in the dynamic and lateral pressure if the lumen of the tube changes. If, as shown in Fig. 3, the lumen of the tube decreases as the fluid progresses from section 1 to section 2 of the tube, then, according to the equation of continuity (3), the average speed of the fluid flow in section 2 increases. As a result, the dynamic pressure increases and, according to

Bernoulli's equation, the lateral pressure must decrease ($h_D < h_B$) so that the total pressure remains unchanged (compare the heights of fluid columns in the manometric tubes A, B, C and D in Fig. 3).

For the real fluids (i.e. fluids showing non-zero viscosity), the validity of Bernoulli's equation is limited. The total pressure is no longer constant but progressively decreases down the direction of flow (Fig. 4). It is a result of the loss of energy of the flowing fluid, due to viscous friction. Notwithstanding, Bernoulli's equation permits correct predictions of concerning changes in pressure along short segments of the tube or vessel, for instance in blood vessels at places of physiological or pathological constriction or extension. For example, constriction of coronary vessels results in the increase in speed of blood flow; the dynamic pressure increases and, in consequence, the lateral pressure decreases. As the lateral pressure decreases the external pressure exerted by surrounding tissues may lead to closure of the vessels and consequently to the ischemic heart stroke. A similar situation is possible as a result of narrowing of the cervical arteries caused by cholesterol deposits accumulation on their walls. The narrowing leads again to an increase in the dynamic pressure, thus to a decrease in the lateral pressure, which poses a threat of the artery closure that can be followed by the ischemic brain stroke.

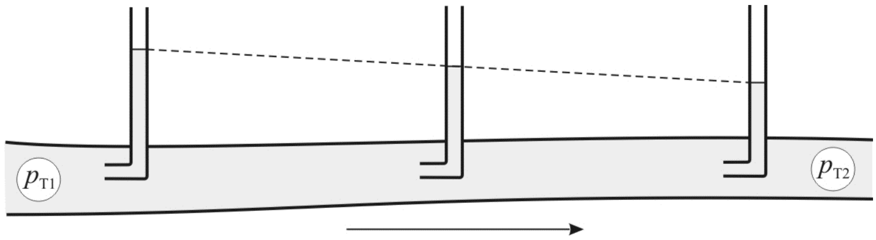

Fig. 4. The flow of real fluid is accompanied by unavoidable drop in the total pressure $\Delta p_T = p_{T1} - p_{T2}$ along the direction of flow . Thus, to maintain the constant rate of flow a difference in pressures between tube ends in necessary.

4. Hagen-Poiseuille Law; Vascular Resistance

As mentioned above, to maintain a steady flow of any real fluid, a pressure difference Δp between the ends of the tube, (blood vessel) is necessary. Moreover, if:

- the vessel's walls are stiff,
- the fluid is incompressible,
- the flow is laminar, and
- the vessel's cross-section is circular,

the rate of fluid flow Q equals:

$$Q = \frac{\pi \cdot r^4}{8 \cdot \eta \cdot l} \cdot \Delta p \tag{12}$$

where: r - the radius of the tube in which the fluid flows,
l - the length of the tube,
η - the viscosity of fluid.

The above formula is known as the *Hagen-Poiseuille law.* As can be noticed, the rate of flow is directly proportional to the difference in pressures between the ends of the tube ($Q \propto \Delta p$), whereas the factor:

$$\frac{\pi \cdot r^4}{8 \cdot \eta \cdot l} \tag{13}$$

is the proportionality coefficient between Q and Δp. It is interpreted by analogy to the Ohm's law.
To do this we need to recall the Ohm's law. It concerns the relationship between voltage and current in an ideal conductor. This relationship states that the intensity of the electric current i flowing through

the ideal conductor of resistance R is proportional to the potential difference, ΔV (voltage), across the conductor:

$$i = \frac{\Delta V}{R} \tag{14}$$

The rate of flow $Q = \frac{\Delta V}{\Delta t}$ is the analogue of the electric current intensity $i = \frac{\Delta q}{\Delta t}$, the difference in pressures Δp causing the flow of fluid is the analogue of the potential difference ΔV existing between the ends of a resistor and causing the flow of the electric charge Δq. Thus, the factor $\frac{\pi \cdot r^4}{8 \cdot \eta \cdot l}$ is the analogue of the reciprocal resistance:

$$\frac{1}{R} = \frac{\pi \cdot r^4}{8 \cdot \eta \cdot l} \tag{15}$$

Adding a subscript V to the symbol of resistance, i.e. R_V, we obtain the expression for the so-called *vascular resistance*:

$$R_V = \frac{8 \cdot \eta \cdot l}{\pi \cdot r^4} \tag{16}$$

The Hagen-Poiseuille law can be now rewritten as follows:

$$Q = \frac{1}{R_V} \cdot \Delta p \tag{17}$$

The Hagen-Poiseuille law is applied for description of functioning of the circulatory system. For instance it permits calculation of the rate of blood flow Q through a particular organ of known vascular resistance R_V, if the difference in pressure Δp between the main organ's afferent artery and in the main organ's efferent vein is known. An increase in Δp with unchanged Q reveals an increase in vascular resistance which is a manifestation of functional or pathological changes within the organ. The flow rate dependence on the factor r^4 (the fourth power of the tube's radius) as given in equations 12, 13 or 16 is of great importance from the point of view of human physiology. Even small changes in the blood vessel's lumen (in the case of arteries caused by contraction of coat of smooth muscles) leads to a significant change in the flow rate, on condition of unchanged Δp. For instance if the radius r is doubled, the flow rate is increased by a factor of 16, and conversely, if the radius is halved, the flow rate is decreased by a factor of 16. Similarly, changes (even small ones) in the radii of superficial skin vessels significantly change the amount of blood circulating in the skin, which our organism uses to control the body temperature.

There are two distinct types of flow, the *laminar flow* and the *turbulent flow*. In the laminar flow of a viscous fluid, particles of the fluid move in parallel layers, each of which has different but constant speed (Fig. 5). Stream lines do not cross one another, the flow is smooth. Above a certain critical speed, the fluid flow becomes turbulent. The turbulent flow is irregular and is characterized by small whirlpool-like regions, called eddy currents.

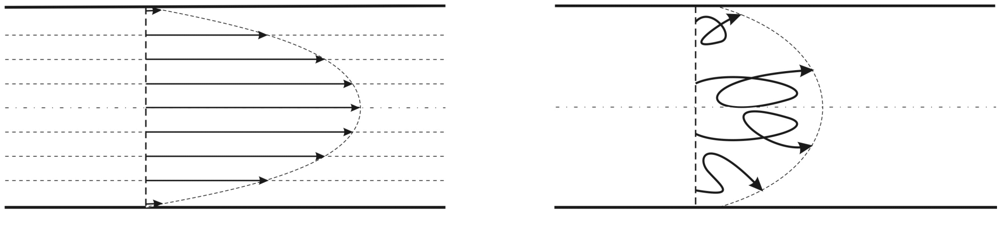

Fig. 5. The laminar flow with its characteristic, parabolic velocity profile and smooth stream lines (left) and the turbulent flow with whirlpool-like patterns and flattened velocity profile (right).

It is accompanied by much greater loss of energy than the laminar flow. Moreover, besides the loss of energy needed to overcome viscous friction, some energy is changed into that of acoustic waves. These waves can be used in diagnostics when auscultating the heart, or bronchial tubes using a stethoscope. Also during measurements of blood pressure by a sphygmomanometer, the acoustic waves accompanying the turbulent flow are used. As the pressure in the cuff of a sphygmomanometer occluding the brachial artery falls slightly below the systolic blood pressure, the stethoscope permits detection of a pounding sounds when blood start to flow again in the artery.

The type of flow can be roughly predicted by the value of the dimensionless quantity called the *Reynold's number Re*:

$$Re = \frac{2 \cdot r \cdot v \cdot \rho}{\eta} \tag{18}$$

where: r - the radius of the tube in which the fluid flows,
v - the average speed of the fluid,
ρ - the fluid density,
η - the fluid viscosity.

Experimental observations show that for tubes of circular cross-section, if:

$Re < 2000$ – the flow is laminar,

$2000 < Re < 3000$ – laminar and turbulent flows are possible; the flow is described as transition type, and it can change from the laminar to turbulent depending on such factors as the pipe roughness and/or flow uniformity,

$Re > 3000$ – the flow is mainly turbulent.

The Reynolds number *Re* shows that for a fluid of given density ρ, the tendency for turbulent flow increases with increasing average speed of flow v and increasing radius r of the tube, whereas the increasing viscosity η reduces this tendency.

5. Experimental procedure

5.1. Determination of the dependence of the vascular resistance and the average speed of flow on the tube radius.

1. Set the rate of flow Q and the viscosity η of blood so that the flow is laminar (typical values for the human circulatory system at rest are: $Q = 5 \div 6$ dm^3/min, $\eta = 2 \div 4 \cdot 10^{-3}$ Pa·s).
2. Change the radius of the tube within the values from 10.0 mm to 12.5 mm. For every successive value of the radius read the value of the total pressure p_T at the beginning of the tube and values of the lateral pressure p_{L1} and p_{L2} at the beginning and at the end of the tube (values of pressures are displayed at proper manometric tubes on the monitor screen).
3. Calculate changes in the lateral pressure:
$$\Delta p = p_{L1} - p_{L2}$$
4. On the basis of equation (17) calculate values of the vascular resistance R_V.
5. From equation (11) and from already recorded values of the total and lateral pressure, p_T and p_{L1} respectively, calculate the values of the dynamic pressure p_d.
6. On the basis of eq. (4) calculate the values of the average speed of flow, taking the density of blood equal to 1060 kg/m^3.
7. For one of the value of r (point 3 of the laboratory report) chose proper values of the rate of flow Q and the viscosity η of blood so that $Re > 3000$ and calculate the average speed of flow v_t in the conditions of the turbulent flow. Plot out the point (v_t, r) on the graph of $v = f(r)$ (see point 8).
8. Make graphs of dependences:
   - $R_V = f\left(\frac{1}{r^4}\right)$ - the vascular resistance vs. inverse of the radius of the tube raised to the power of 4,
   - $v = f(r)$ - the average speed of flow vs. the radius of the tube.

5.2. Determination of the dependence of the average speed of flow on the rate of flow.

1. Set a value of the tube radius from 10.0 mm to 12.5 mm interval and the viscosity η of blood from 2 to 4×10^{-3} Pa·s so that the flow is laminar (!).
2. Change values of the rate of flow Q and every time, for successive value of the rate of flow, read value of the total pressure p_T and values of the lateral pressure p_{L1} and p_{L2} at both ends of the tube (values of pressures are displayed at proper manometric tubes on the monitor screen). Calculate changes in the lateral pressure $\Delta p = p_{L1} - p_{L2}$.
3. From formula (17) calculate values of the vascular resistance R_V.
4. Calculate the values of the dynamic pressure p_{d1} in section 1 of the vessel using formula (11).
5. Calculate the values of the average speed of blood flow in section 1 of the vessel using formula (4), taking the density of blood as $\rho_b = 1060$ kg/m^3.
6. For one of values of Q (point 3 of the laboratory report), choose values of the tube radius r and the viscosity η of the blood so that $Re > 3000$ and calculate the average speed of flow v_t in the conditions of turbulent flow.
7. Make a graph $v = f(Q)$ of the average speed of flow versus the rate of flow.